

# Counting conjugacy classes of subgroups in a finitely generated group \*

Alexander Mednykh

*Sobolev Institute of Mathematics, Novosibirsk State University,  
630090, Novosibirsk, Russia  
mednykh@math.nsc.ru*

## Abstract

A new general formula for the number of conjugacy classes of subgroups of given index in a finitely generated group is obtained.

Keywords: *number of subgroups, conjugacy class of subgroups, surface coverings*

## 1 Introduction

Let  $M_\Gamma(n)$  denote the number of subgroups of index  $n$  in a group  $\Gamma$ , and  $N_\Gamma(n)$  be the number of conjugacy classes of such groups. The last function counts the isomorphism classes of transitive permutation representations of degree  $n$  of  $\Gamma$  and hence, also the equivalence classes of  $n$ -fold unbranched connected coverings of a topological space with fundamental group  $\Gamma$ .

Each subgroup  $K$  of index  $n$  in a group  $\Gamma$  determines a transitive action of degree  $n$  of  $\Gamma$  on the cosets of  $K$ , and each transitive action of  $\Gamma$  on a set of cardinality  $n$  is isomorphic to such an action, with  $K$  uniquely determined up to conjugacy in  $\Gamma$ . Starting with such a consideration M. Hall [3] determined the numbers of subgroups  $M_\Gamma(n)$  for a free group  $\Gamma = F_r$  of rank  $r$ . Later V. Liskovets [7] developed a new method for calculation of  $N_\Gamma(n)$  for the same group. Both functions  $M_\Gamma(n)$  and  $N_\Gamma(n)$  for the fundamental group  $\Gamma$  of a closed surface were obtained in [12] and [13] for orientable and non-orientable surfaces, respectively. See also [14] and [2] for the case of the fundamental group of the Klein bottle and a survey [6] for related problems. In all these cases the problem of calculation of  $M_\Gamma(n)$  was solved essentially due to the ideas by Hurwitz and Frobenius to contribute the representation theory of symmetric groups as the main tool ([4], [5]). The solution for the problem to finding  $N_\Gamma(n)$  was based on the further development of the Liskovets method ([7], [8]). In [9] and [10], these ideas were applied to determine  $M_\Gamma(n)$  for the

---

\*Supported by the RFBR (grant 03-01-00104) and by INTAS (grant 03-51-3663)

fundamental groups of some Seifert spaces. Asymptotic formulas for  $M_\Gamma(n)$  in many important cases were obtained in series of papers by T. W. Müller and his collaborators ([16],[17],[18]). An excellent exposition of the above results and related subjects is given in the book [11].

In the present paper, a new formula for the number of conjugacy classes of subgroups of given index in a finitely generated group is obtained. For the sake of simplicity, we require the group  $\Gamma$  to be finitely generated. But, indeed, all results of the paper remain to be true for the groups having only finitely many subgroups of each finite index (which is always the case for finitely generated groups).

The main counting principle suggested in Section 2 of the paper is rather universal. It can be applied to Fuchsian groups to calculate the number of non-equivalent surface coverings (Section 3) as well as the number of unrooted maps on the surface [15]. In Section 4, some general approach is developed to find the number of non-equivalent unbranched coverings of a manifold with finitely generated fundamental group.

## 2 The main counting principle

Denote by  $\text{Epi}(K, \mathbb{Z}_\ell)$  the set of epimorphisms of a group  $K$  onto the cyclic group  $\mathbb{Z}_\ell$  of order  $\ell$  and by  $|E|$  the cardinality of a set  $E$ .

The main result of this paper is the following counting principle.

**Theorem 1** *Let  $\Gamma$  be a finitely generated group. Then the number of conjugacy classes of subgroups of index  $n$  in the group  $\Gamma$  is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{\substack{K < \Gamma \\ m}} |\text{Epi}(K, \mathbb{Z}_\ell)|,$$

where the sum  $\sum_{\substack{K < \Gamma \\ m}}$  is taken over all subgroups  $K$  of index  $m$  in the group  $\Gamma$ .

**Proof:** Let  $P$  be a subgroup in  $\Gamma$  and  $N(P, \Gamma)$  is the normalizer of  $P$  in the group  $\Gamma$ . We need the following two elementary lemmas.

**Lemma 1** *The number of conjugacy classes of subgroups of index  $n$  in the group  $\Gamma$  is given by the formula*

$$N_\Gamma(n) = \frac{1}{n} \sum_{\substack{P < \Gamma \\ n}} |N(P, \Gamma)/P|.$$

**Proof:** Let  $E$  be a conjugacy class of subgroups of index  $n$  in the group  $\Gamma$ . We claim that

$$\sum_{P \in E} |N(P, \Gamma)/P| = n.$$

Indeed, let  $P' \in E$ . Then  $|E| = |\Gamma : N(P', \Gamma)|$  and for any  $P \in E$  the groups  $N(P, \Gamma)/P$  and  $N(P', \Gamma)/P'$  are isomorphic. We have

$$\sum_{P \in E} |N(P, \Gamma)/P| = |E| |N(P', \Gamma)/P'| = |\Gamma : N(P', \Gamma)| |N(P', \Gamma) : P'| = |\Gamma : P'| = n.$$

Hence,

$$n N(n) = \sum_E n = \sum_E \sum_{P \in E} |N(P, \Gamma)/P| = \sum_{P \leq_n \Gamma} |N(P, \Gamma)/P|,$$

where the sum  $\sum_E$  is taken over all conjugacy classes  $E$  of subgroups of index  $n$  in the group  $\Gamma$ .  $\square$

**Lemma 2** *Let  $P$  be a subgroup of index  $n$  in the group  $\Gamma$ . Then*

$$|N(P, \Gamma)/P| = \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\substack{P \triangleleft K < \Gamma \\ Z_\ell \leq m}} \phi(\ell),$$

where  $\phi(\ell)$  is the Euler function and the second sum is taken over all subgroups  $K$  of index  $m$  in  $\Gamma$  containing  $P$  as a normal subgroup with  $K/P \cong Z_\ell$ . The sum vanishes if there are no such subgroups.

Proof: Set  $G = N(P, \Gamma)/P$ . Since  $P \triangleleft N(P, \Gamma) < \Gamma$  and  $P \leq_n \Gamma$ , the order of any cyclic subgroup of  $G$  is a divisor of  $n$ .

Note that there is a one-to-one correspondence between cyclic subgroups  $Z_\ell$  in the group  $G$  and subgroups  $K$  satisfying  $P \triangleleft K < \Gamma$ , where  $\ell m = n$ .

Given a cyclic subgroup  $Z_\ell < G$  there are exactly  $\phi(\ell)$  elements of  $G$  which generate  $Z_\ell$ .

Hence,

$$|G| = \sum_{\ell | n} \phi(\ell) \sum_{Z_\ell < G} 1 = \sum_{\ell | n} \phi(\ell) \sum_{\substack{P \triangleleft K < \Gamma \\ Z_\ell \leq m}} 1 = \sum_{\ell | n} \sum_{\substack{P \triangleleft K < \Gamma \\ Z_\ell \leq m}} \phi(\ell).$$

$\square$

We finish the proof of the theorem by applying Lemma 1 and Lemma 2 for  $\ell m = n$  :

$$\begin{aligned} n N(n) &= \sum_{P \leq_n \Gamma} |N(P, \Gamma)/P| = \\ &= \sum_n \sum_{\ell | n} \sum_{\substack{P \triangleleft K < \Gamma \\ Z_\ell \leq m}} \phi(\ell) = \sum_{\ell | n} \sum_{P \leq_n \Gamma} \sum_{\substack{P \triangleleft K < \Gamma \\ Z_\ell \leq m}} \phi(\ell) = \\ &= \sum_{\ell | n} \sum_{K \leq \Gamma} \sum_{\substack{P \triangleleft K \\ Z_\ell \leq m}} \phi(\ell) = \sum_{\ell | n} \sum_{K \leq \Gamma} |\text{Epi}(K, Z_\ell)|. \end{aligned}$$

The last equality is a consequence of the following observation. Given subgroup  $P$ ,  $P \trianglelefteq_{Z_\ell} K$  there are exactly  $\phi(\ell)$  homomorphisms  $\psi : K \rightarrow Z_\ell$ , with  $\text{Ker}(\psi) = P$ .  $\square$

Denote by  $\text{Hom}(\Gamma, Z_\ell)$  the set of homomorphisms of a group  $\Gamma$  into the cyclic group  $Z_\ell$  of order  $\ell$ . Since  $|\text{Hom}(\Gamma, Z_\ell)| = \sum_{d|\ell} |\text{Epi}(\Gamma, Z_d)|$ , by the Möbius inversion formula we have the following result

**Lemma 3 (G. Jones [1])**

$$|\text{Epi}(\Gamma, Z_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) |\text{Hom}(\Gamma, Z_d)|,$$

where  $\mu(n)$  is the Möbius function.

This lemma essentially simplifies the calculation of  $|\text{Epi}(\Gamma, Z_\ell)|$  for a finitely generated group  $\Gamma$ . Indeed, let  $H_1(\Gamma) = \Gamma/[\Gamma, \Gamma]$  be the first homology group of  $\Gamma$ . Suppose that  $H_1(\Gamma) = Z_{m_1} \oplus Z_{m_2} \oplus \dots \oplus Z_{m_s} \oplus Z^r$ . Then we have

**Lemma 4**

$$|\text{Epi}(\Gamma, Z_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) (m_1, d) (m_2, d) \dots (m_s, d) d^r,$$

where  $(m, d)$  is the greatest common divisor of  $m$  and  $d$ .

Proof: Note that  $|\text{Hom}(Z_m, Z_d)| = (m, d)$  and  $|\text{Hom}(Z, Z_d)| = d$ . Since the group  $Z_d$  is Abelian, we obtain

$$|\text{Hom}(\Gamma, Z_d)| = |\text{Hom}(H_1(\Gamma), Z_d)| = (m_1, d) (m_2, d) \dots (m_s, d) d^r.$$

Then the result follows from Lemma 3.  $\square$

In particular, we have

### Corollary 1

- (i) Let  $F_r$  be a free group of rank  $r$ . Then  $H_1(F_r) = \mathbb{Z}^r$  and
- $$|\text{Epi}(F_r, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^r.$$
- (ii) Let  $\Phi_g = \langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] = 1 \rangle$  be the fundamental group of a closed orientable surface of genus  $g$ . Then  $H_1(\Phi_g) = \mathbb{Z}^{2g}$  and
- $$|\text{Epi}(\Phi_g, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2g}.$$
- (iii) Let  $\Lambda_p = \langle a_1, a_2, \dots, a_p : \prod_{i=1}^p a_i^2 = 1 \rangle$  be the fundamental group of a closed non-orientable surface of genus  $p$ . Then  $H_1(\Lambda_p) = \mathbb{Z}_2 \oplus \mathbb{Z}^{p-1}$  and
- $$|\text{Epi}(\Lambda_p, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) (2, d) d^{p-1}.$$

Note that the fundamental group of any compact surface (orientable or not, possibly, with non-empty boundary) is one of the three groups  $F_r$ ,  $\Phi_g$  and  $\Lambda_p$  listed in Corollary 1. In the next two sections we identify the number of conjugacy classes of subgroups of index  $n$  in the group  $\Gamma$  and the number of equivalence classes of  $n$ -fold unbranched connected coverings of a manifold with fundamental group  $\Gamma$ .

## 3 Counting surface coverings

Recall that the fundamental group  $\pi_1(\mathcal{B})$  of a bordered surface  $\mathcal{B}$  of Euler characteristic  $\chi = 1 - r$ ,  $r \geq 0$ , is a free group  $F_r$  of rank  $r$ . An example of such a surface is the disc  $\mathcal{D}_r$  with  $r$  holes. As the first application of the counting principle (Theorem 1) we have the following result obtained earlier by V. Liskovets [7]

**Theorem 2** *Let  $\mathcal{B}$  be a bordered surface with the fundamental group  $\pi_1(\mathcal{B}) = F_r$ . Then the number of non-equivalent  $n$ -fold coverings of  $\mathcal{B}$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{(r-1)m+1} M(m),$$

where  $M(m)$  is the number of subgroups of index  $m$  in the group  $F_r$ .

Proof: Note that all subgroups of index  $m$  in  $F_r$  are isomorphic to  $\Gamma_m = F_{(r-1)m+1}$ . By Theorem 1 we have

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} |\text{Epi}(\Gamma_m, \mathbb{Z}_\ell)| \cdot M(m),$$

By Corollary 1(i) we get

$$|\text{Epi}(\Gamma_m, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{(r-1)m+1}$$

and the result follows.  $\square$

By the M. Hall recursive formula [3] the number of subgroups of index  $m$  in the group  $F_r$  is equal to  $M(m) = \frac{t_{m,r}}{(m-1)!}$ , where

$$t_{m,r} = m!^r - \sum_{j=1}^{m-1} \binom{m-1}{j-1} (m-j)!^r t_{j,r}, \quad t_{1,r} = 1.$$

The next result was obtained in [12] in a rather complicated way

**Theorem 3** *Let  $\mathcal{S}$  be a closed orientable surface with the fundamental group  $\pi_1(\mathcal{S}) = \Phi_g$ . Then the number of non-equivalent  $n$ -fold coverings of  $\mathcal{S}$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2(g-1)m+2} M(m),$$

where  $M(m)$  is the number of subgroups of index  $m$  in the group  $\Phi_g$ .

Proof: In this case, by the Riemann-Hurwitz formula all subgroups of index  $m$  in  $\Phi_g$  are isomorphic to the group  $K_m = \Phi_{(g-1)m+1}$ . By the main counting principle we have

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} |\text{Epi}(K_m, \mathbb{Z}_\ell)| \cdot M(m),$$

where

$$|\text{Epi}(K_m, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2(g-1)m+2}$$

is given by Corollary 1(ii).  $\square$

Let  $\mathcal{N}$  be a closed non-orientable surface of genus  $p$  with the fundamental group  $\pi_1(\mathcal{N}) = \Lambda_p$ . Denote by  $\mathcal{N}_m^+$  and  $\mathcal{N}_m^-$  an orientable and non-orientable  $m$ -fold coverings

of  $\mathcal{N}$ , respectively and set  $\Gamma_m^+ = \pi_1(\mathcal{N}_m^+)$  and  $\Gamma_m^- = \pi_1(\mathcal{N}_m^-)$ . For simplicity, we will refer to  $\Gamma_m^+$  and  $\Gamma_m^-$  as orientable and non-orientable subgroups of index  $m$  in  $\Lambda_p$ , respectively. By the Riemann-Hurwitz formula we get

$$2 \text{ genus}(\mathcal{N}_m^+) - 2 = m(p-2) \text{ and } \text{genus}(\mathcal{N}_m^-) - 2 = m(p-2),$$

where  $p = \text{genus}(\mathcal{N})$ . Hence  $\Gamma_m^+ = \Phi_{\frac{m}{2}(p-2)+1}$  and  $\Gamma_m^- = \Lambda_{m(p-2)+2}$ .

By the main counting principle, the number of non-equivalent  $n$ -fold coverings of  $N$  is given by the formula

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} (|\text{Epi}(\Gamma_m^+, \mathbb{Z}_\ell)| \cdot M^+(m) + |\text{Epi}(\Gamma_m^-, \mathbb{Z}_\ell)| \cdot M^-(m)),$$

where  $M^+(m)$  and  $M^-(m)$  are the numbers of orientable and non-orientable subgroups of index  $m$  in the group  $\Lambda_p$ , respectively.

By Corollary 1(ii) and Corollary 1(iii), we have

$$|\text{Epi}(\Gamma_m^+, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^{m(p-2)+2} \text{ and } |\text{Epi}(\Gamma_m^-, \mathbb{Z}_\ell)| = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) (2, d) d^{m(p-2)+1}.$$

As a result, we have proved the following theorem obtained earlier in [13] by making use of a cumbersome combinatorial technique.

**Theorem 4** *Let  $n$  be a closed orientable surface with the fundamental group  $\pi_1(\mathcal{N}) = \Lambda_p$ . Then the number of non-equivalent  $n$ -fold coverings of  $\mathcal{N}$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell|n \\ \ell m=n}} \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) (d^{m(p-2)+2} M^+(m) + (2, d) d^{m(p-2)+1} M^-(m)),$$

where  $M^+(m)$  and  $M^-(m)$  are the numbers of orientable and non-orientable subgroups of index  $m$  in the group  $\Lambda_p$ , respectively.

For completeness note that ([12], [13]) if  $\Gamma = \Phi_g$  or  $\Lambda_p$  then the number  $M(m)$  of subgroups of index  $m$  in the group  $\Gamma$  is equal to

$$R_\nu(m) = m \sum_{s=1}^m \frac{(-1)^{s+1}}{s} \sum_{\substack{i_1+i_2+\dots+i_s=m \\ i_1, i_2, \dots, i_s \geq 1}} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_s},$$

where  $\beta_k = \sum_{\chi \in D_k} \left(\frac{k!}{f^\chi}\right)^\nu$ ,  $D_k$  is the set of irreducible representations of a symmetric group  $S_k$ ,  $f^\chi$  is the degree of the representation  $\chi$ ,  $\nu = 2g - 2$  for  $\Gamma = \Phi_g$  and  $\nu = p - 2$  for  $\Gamma = \Lambda_p$ . Moreover, in the latter case,  $M^+(m) = 0$  if  $m$  is odd,  $M^+(m) = R_{2\nu}(\frac{m}{2})$  if  $m$  is even, and  $M^-(m) = M(m) - M^+(m)$ . Also, the number of subgroups can be found by the following recursive formula

$$M(m) = m \beta_m - \sum_{j=1}^{m-1} \beta_{m-j} M(j), \quad M(1) = 1.$$

## 4 Non-equivalent coverings of manifolds

All manifolds in this section are supposed to be connected, with finitely generated fundamental group. No restriction on dimension is given. The manifolds under consideration can be closed, open or bordered, orientable or not. The following theorem is just a topological version of Theorem 1.

**Theorem 5** *Let  $M$  be a connected manifold with finitely generated fundamental group  $\Gamma = \pi_1(M)$ . Then the number of non-equivalent  $n$ -fold coverings of  $M$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\Phi \in \mathfrak{F}_m} |\text{Epi}(\Phi, Z_\ell)| \cdot M_{\Phi, \Gamma}(m),$$

where  $\mathfrak{F}_m$  is the set of groups arising as fundamental groups of  $m$ -fold coverings of  $M$  and  $M_{\Phi, \Gamma}(m)$  is the number of subgroups of index  $m$  in the group  $\Gamma$  that are isomorphic to  $\Phi$ .

Taking into account Lemma 3, we obtain the following result as an immediate corollary of Theorem 5

**Theorem 6** *Let  $M$  be a connected manifold with finitely generated fundamental group  $\Gamma = \pi_1(M)$ . Then the number of non-equivalent  $n$ -fold coverings of  $M$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{\Phi \in \mathfrak{F}_m} \sum_{d | \ell} \mu\left(\frac{\ell}{d}\right) |\text{Hom}(\Phi, Z_d)| \cdot M_{\Phi, \Gamma}(m),$$

where  $\mathfrak{F}_m$  is the set of groups arising as fundamental groups of  $m$ -fold coverings of  $M$  and  $M_{\Phi, \Gamma}(m)$  is the number of subgroups of index  $m$  in the group  $\Gamma$  that are isomorphic to  $\Phi$ .

Let  $M$  be a manifold and  $\Gamma = \pi_1(M)$ . Denote by  $H_1(M)$  and  $H_1(\Gamma)$  the first homology over  $Z$  of the manifold  $M$  and the group  $\Gamma$ , respectively. Recall that  $H_1(\Gamma) = \Gamma/[\Gamma, \Gamma]$  and  $H_1(M) = H_1(\Gamma)$ .

Since the group  $Z_d$  is Abelian, there is a one-to-one correspondence between the sets  $\text{Hom}(\Gamma, Z_d)$  and  $\text{Hom}(H_1(\Gamma), Z_d)$ . As a result we have the following homological version of the previous theorem

**Theorem 7** *Let  $M$  be a connected manifold with finitely generated fundamental group  $\Gamma = \pi_1(M)$ . Then the number of non-equivalent  $n$ -fold coverings of  $M$  is given by the formula*

$$N(n) = \frac{1}{n} \sum_{\substack{\ell | n \\ \ell m = n}} \sum_{H \in \mathfrak{H}_m} \sum_{d | \ell} \mu\left(\frac{\ell}{d}\right) |\text{Hom}(H, Z_d)| \cdot M'_{H, \Gamma}(m),$$



where  $\mathfrak{H}_m$  is the set of groups arising as homologies of  $m$ -fold coverings of  $M$  and  $M'_{H,\Gamma}(m)$  is the number of subgroups  $F$  of index  $m$  in the group  $\Gamma$ , with  $H_1(F)$  isomorphic to  $H$ .

**Acknowledgement:** The author gratefully thanks Com<sup>2</sup>MaC, Pohang University of Science and Technology (Korea) for hospitality during preparation of the present paper.

## References

- [1] G. A. Jones, Enumeration of homomorphisms and surface coverings, *Quart. J. Math. Oxford* **46**:2 (1995), 485–507.
- [2] G. A. Jones and V. Liskovets, Enumerating of subgroups of finite index, In: *Mini-Workshop on Hurwitz Theory and Ramifications* (ed. by J. H. Kwak et A. D. Mednykh), Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology, 2003, 185–188.
- [3] M. Hall, Jr., Subgroups of finite index in free groups, *Canad. J. Math.*, **1**, No.1 (1949), 187–190.
- [4] A. Hurwitz, Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **39** (1891), 1–60.
- [5] A. Hurwitz, Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **55** (1902), 53–66.
- [6] J. H. Kwak and J. Lee, Enumeration of graph coverings, surface branched coverings and related group theory, In: *Combinatorial and Computational Mathematics: Present and Future* (ed. by S. Hong, J. H. Kwak et al.), Word Scientific, Singapore, 2001, 97–161.
- [7] V. Liskovets, Towards the enumeration of subgroups of a free group, *Dokl. Akad. Nauk BSSR* **15** (1971), 6–9 (in Russian).
- [8] V. Liskovets, Recursive enumeration under mutually orthogonal group action, *Acta-Appl. Math.* **52** (1998), 91–120.
- [9] V. Liskovets and A. Mednykh, Enumeration of subgroups in the fundamental groups of orientable circle bundles over surfaces, *Comm. in Algebra*, **28**, No.4, (2000), 1717–1738.
- [10] V. Liskovets and A. Mednykh, The number of subgroups in the fundamental groups of some non-oriented 3-manifolds, In: *Formal power series and algebraic combinatorics. Proceedings of the 12th international conference, FPSAC'00, Moscow, Russia, June 26-30, 2000*, Springer, Berlin, 2000, 276–287.

- [11] A.Lubotzky and D.Segal, Subgroup Growth, Birkhäuser Verlag, Basel-Boston-Berlin, 2003.
- [12] A.D. Mednykh, Hurwitz problem on the number of nonequivalent coverings of a compact Riemann surface, *Siber. Math. J.* **23**, No.3, (1982), 415-420.
- [13] A.D. Mednykh and G.G. Pozdnyakova, Number of nonequivalent coverings over a nonorientable compact surface, *Siber. Math. J.* **27**, No.1, (1986), 99-106.
- [14] A.D. Mednykh, On the number of subgroups in the fundamental group of a closed surfaces, *Comm. in Algebra*, **16**, No.10, (1988), 2137–2148.
- [15] A.D. Mednykh and R. Nedela, Enumeration of unrooted maps with given genus (2004), to appear
- [16] T. W. Müller, Enumerating representations in finite wreath products, *Adv. Math.*, **153**, No.1, (2000), 118–154.
- [17] T. W. Müller and J. Shareshian, Enumerating representations in finite wreath products. II: Explicit formulae, *Adv. Math.*, **171**, No.2, (2002), 276–331.
- [18] T. W. Müller and J.-C. Puchta, Character theory of symmetric groups and subgroup growth of surface groups, *J. London Math. Soc. (2)*, **66** (2002), 623–640.